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On new Fejér type inequalities for $m-$convex and quasi convex functions

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Abstract

In this paper we establish new inequalities of weighted version of Hermite-Hadamard type inequality for functions whose derivatives absolute values are $m-$ convex. Also we obtain some Fejér type inequalities for quasi-convex functions.

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1 Introduction

The following double inequality is well known in the literature as Hadamard's inequality:

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on an interval I of real numbers, $a, b \in I$ and $a < b$, we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave.

It was first discovered by Hermite in 1881 in the Journal Mathesis (see [10]). The inequality (1.1) was nowhere mentioned in the mathematical literature until 1893. Beckenbach, a leading expert on the theory of convex functions, wrote that inequality (1.1) was proven by Hadamard in 1893 (see [11]). In 1974 Mitrinovič found Hermite's note in Mathesis. That is why, the inequality (1.1) was known as Hermite-Hadamard inequality.

In [1], Fejér established the following Fejér inequality which is the weighted generalization of Hermite-Hadamard inequality:

Theorem 1.1. Let $f : I \to \mathbb{R}$ be convex on I and let $a, b \in I$ with $a < b$. Then the inequality

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx \tag{1.2}$$

holds, where $g : [a, b] \to \mathbb{R}$ is nonnegative and symmetric to $\frac{a+b}{2}$.

If $g = 1$, then we are talking about the Hermite-Hadamard inequalities. More about those inequalities can be found in a number of papers and monographies. For recent results and generalizations concerning Fejér inequality (1.2) see [2]-[8].

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Definition 1.2. A function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if whenever $x, y \in [a, b]$ and $t \in [0, 1]$, the following inequality holds:

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y).$$

We say that f is concave if $(-f)$ is convex.

This definition has its origins in Jensen's results from [9] and has opened up the most extended, useful and multi-disciplinary domain of mathematics, namely, convex analysis. Convex curves and convex bodies have appeared in mathematical literature since antiquity and there are many important results related to them.

In [12], G. Toader defined $m-$convexity as the following:

Definition 1.3. The function $f : [0, b] \to \mathbb{R}, b > 0$, is said to be $m-$convex, where $m \in [0, 1]$, if we have

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$. Denote by $K_m(b)$ the set of the $m-$convex functions on $[0, b]$ for which $f(0) \le 0$.

In [13], Set *et al.* proved the following inequality of Hermite-Hadamard type for m-convex functions.

Theorem 1.4. Let $f : I^\circ \subset [0, b^*] \to \mathbb{R}, b^* > 0$, be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is $m-$convex on $[a, b]$, $q > 1$ and $m \in (0, 1]$, then the following inequality holds:

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \tag{1.3}$$
$$\le \frac{(b-a)}{4} \left\{ \left(|f'(a)|^q + 3m \left| f'\left(\frac{b}{m}\right) \right|^q \right)^{\frac{1}{q}} + \left(3\,|f'(a)|^q + m \left| f'\left(\frac{b}{m}\right) \right|^q \right)^{\frac{1}{q}} \right\}.$$

In [8], Sarıkaya proved the following Lemmas for Fejér type inequalities:

Lemma 1.5. Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$ and $w : [a, b] \to [0, \infty)$ be a differentiable mapping. If $f' \in L[a, b]$, then the following equality holds:

$$\int_a^b f(x)w(x)dx - f\left(\frac{a+b}{2}\right) \int_a^b w(x)dx = (b-a)^2 \int_0^1 k(t)f'(ta + (1-t)b)dt$$

for each $t \in [0, 1]$, where

$$k(t) = \begin{cases} \int_0^t w(sa + (1-s)b)ds, & t \in \left[0, \frac{1}{2}\right) \\ -\int_t^1 w(sa + (1-s)b)ds, & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Lemma 1.6. Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$ and $w : [a, b] \to [0, \infty)$ be a differentiable mapping. If $f' \in L[a, b]$, then the following equality holds:

$$\frac{f(a)+f(b)}{2}\int_a^b w(x)dx - \int_a^b f(x)w(x)dx = \frac{(b-a)^2}{2}\int_0^1 p(t)f'(ta+(1-t)b)dt$$

for each $t \in [0, 1]$, where

$$p(t) = \int_t^1 w(sa+(1-s)b)ds - \int_0^t w(sa+(1-s)b)ds.$$

The aim of this paper is to establish new inequalities of weighted version of Hermite-Hadamard type inequality for functions whose derivatives absolute values are $m-$ convex. Also we obtain some new Fejér type inequalities for quasi-convex functions.

2 Inequalities for $m-$convex functions

Theorem 2.1. Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$ and $w : [a, b] \to [0, \infty)$ be a differentiable mapping. If $|f'|$ is $m-$convex on $[a, b]$ for some fixed $m \in (0, 1]$ then the following inequality holds:

$$\begin{aligned}
&\left|\int_a^b f(x)w(x)dx - f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx\right| \\
\leq\ & \frac{(b-a)^2}{6}\left\{\|w\|_{[0,\frac{1}{2}],\infty}\left(|f'(a)| + 2m\left|f'\left(\frac{b}{m}\right)\right|\right) + \|w\|_{[\frac{1}{2},1],\infty}\left(2|f'(a)| + m\left|f'\left(\frac{b}{m}\right)\right|\right)\right\} \\
\leq\ & \frac{(b-a)^2}{8}\|w\|_{[0,1],\infty}\left(|f'(a)| + m\left|f'\left(\frac{b}{m}\right)\right|\right).
\end{aligned} \tag{2.1}$$

Proof. From Lemma 1.5, using the properties of modulus, we have

$$\begin{aligned}
&\left|\int_a^b f(x)w(x)dx - f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx\right| \\
\leq\ & (b-a)^2\left\{\int_0^{\frac{1}{2}}\left|\int_0^t w(sa+(1-s)b)ds\right| |f'(ta+(1-t)b)|\,dt\right. \\
&\left. + \int_{\frac{1}{2}}^1\left|\int_t^1 w(sa+(1-s)b)ds\right| |f'(ta+(1-t)b)|\,dt\right\} \\
\leq\ & (b-a)^2\left\{\|w\|_{[0,\frac{1}{2}],\infty}\int_0^{\frac{1}{2}} t\,|f'(ta+(1-t)b)|\,dt\right. \\
&\left. + \|w\|_{[\frac{1}{2},1],\infty}\int_{\frac{1}{2}}^1 (1-t)\,|f'(ta+(1-t)b)|\,dt\right\}.
\end{aligned}$$

Since $|f'|$ is $m-$convex on $[a,b]$, we know that for $t\in[0,1]$

$$|f'(ta+(1-t)b)| = \left|f'(ta+m(1-t)\frac{b}{m})\right| \leq t\,|f'(a)| + m(1-t)\left|f'\left(\frac{b}{m}\right)\right|,$$

hence

$$\begin{aligned}
&\left|\int_a^b f(x)w(x)dx - f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx\right| \qquad (2.2)\\
\leq\ & (b-a)^2\left\{\|w\|_{[0,\frac{1}{2}],\infty}\int_0^{\frac{1}{2}} t\left[t\,|f'(a)| + m(1-t)\left|f'\left(\frac{b}{m}\right)\right|\right]dt\right.\\
&\left.+\|w\|_{[\frac{1}{2},1],\infty}\int_{\frac{1}{2}}^1 (1-t)\left[t\,|f'(a)| + m(1-t)\left|f'\left(\frac{b}{m}\right)\right|\right]dt\right\}\\
\leq\ & \frac{(b-a)^2}{6}\left\{\|w\|_{[0,\frac{1}{2}],\infty}\left(|f'(a)| + 2m\left|f'\left(\frac{b}{m}\right)\right|\right)\right.\\
&\left.+\|w\|_{[\frac{1}{2},1],\infty}\left(2\,|f'(a)| + m\left|f'\left(\frac{b}{m}\right)\right|\right)\right\}.
\end{aligned}$$

Also

$$\|w\|_{[0,\frac{1}{2}],\infty} \leq \|w\|_{[0,1],\infty}$$

and

$$\|w\|_{[\frac{1}{2},1],\infty} \leq \|w\|_{[0,1],\infty}$$

by using (2.2), we obtain (2.1). This completes the proof. ∎

Theorem 2.2. Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a,b \in I^\circ$ with $a<b$ and $w : [a,b] \to [0,\infty)$ be a differentiable mapping. If $|f'|$ is $m-$convex on $[a,b]$, $q>1$, for some fixed $m\in(0,1]$ then the following inequality holds:

$$\begin{aligned}
&\left|\int_a^b f(x)w(x)dx - f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx\right|\\
\leq\ & \frac{(b-a)^2}{4(p+1)^{1/p}}\left\{\|w\|_{[0,\frac{1}{2}],\infty}\left(\frac{|f'(a)|^q + 3m\left|f'\left(\frac{b}{m}\right)\right|^q}{4}\right)^{\frac{1}{q}}\right.\\
&\left.+\|w\|_{[\frac{1}{2},1],\infty}\left(\frac{3\,|f'(a)|^q + m\left|f'\left(\frac{b}{m}\right)\right|^q}{4}\right)^{\frac{1}{q}}\right\}\\
\leq\ & \frac{(b-a)^2}{4(p+1)^{1/p}}\|w\|_{[0,1],\infty}\left\{\left(\frac{|f'(a)|^q + 3m\left|f'\left(\frac{b}{m}\right)\right|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3\,|f'(a)|^q + m\left|f'\left(\frac{b}{m}\right)\right|^q}{4}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Proof. Using Lemma 1.5 and Hölder inequality, we obtain

$$\begin{aligned}
&\left|\int_a^b f(x)w(x)dx - f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx\right| \\
\leq\ & (b-a)^2\left\{\int_0^{\frac{1}{2}}\left|\int_0^t w(sa+(1-s)b)ds\right|\left|f'(ta+(1-t)b)\right|dt\right. \\
&\left.+\int_{\frac{1}{2}}^1\left|\int_t^1 w(sa+(1-s)b)ds\right|\left|f'(ta+(1-t)b)\right|dt\right\} \\
\leq\ & (b-a)^2\left\{\left(\int_0^{\frac{1}{2}}\left|\int_0^t w(sa+(1-s)b)ds\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^1\left|\int_t^1 w(sa+(1-s)b)ds\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right\} \\
\leq\ & (b-a)^2\left\{\|w\|_{[0,\frac{1}{2}],\infty}\left(\int_0^{\frac{1}{2}} t^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\|w\|_{[\frac{1}{2},1],\infty}\left(\int_{\frac{1}{2}}^1 |1-t|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

for $\frac{1}{p}+\frac{1}{q}=1$. Since $|f'|^q$ is $m-$convex on $[a,b]$, we have

$$\begin{aligned}
&\left|\int_a^b f(x)w(x)dx - f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx\right| \\
\leq\ & \frac{(b-a)^2}{4(p+1)^{1/p}}\left\{\|w\|_{[0,\frac{1}{2}],\infty}\left(\int_0^{\frac{1}{2}}\left[t\left|f'(a)\right|^q+m(1-t)\left|f'\left(\frac{b}{m}\right)\right|^q\right]dt\right)^{\frac{1}{q}}\right. \\
&\left.+\|w\|_{[\frac{1}{2},1],\infty}\left(\int_{\frac{1}{2}}^1\left[t\left|f'(a)\right|^q+m(1-t)\left|f'\left(\frac{b}{m}\right)\right|^q\right]dt\right)^{\frac{1}{q}}\right\} \\
=\ & \frac{(b-a)^2}{4(p+1)^{1/p}}\left\{\|w\|_{[0,\frac{1}{2}],\infty}\left(\frac{|f'(a)|^q+3m\left|f'\left(\frac{b}{m}\right)\right|^q}{4}\right)^{\frac{1}{q}}\right. \\
&\left.+\|w\|_{[\frac{1}{2},1],\infty}\left(\frac{3\left|f'(a)\right|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q}{4}\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

Also

$$\int_0^{\frac{1}{2}} t^p dt = \int_{\frac{1}{2}}^1 (1-t)^p dt = \frac{1}{2^{p+1}(p+1)}.$$

This completes the proof. ∎

Remark 2.3. Since $\left(\frac{1}{p+1}\right)^{\frac{1}{p}} < 1$ and $\frac{1}{4^{1/q}} < 1$, if we choose $w(x) = 1$ in Theorem 2.2, we obtain the inequalties (1.3).

Theorem 2.4. Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$ and $w : [a, b] \to [0, \infty)$ be a differentiable mapping. If $|f'|$ is $m-$convex on $[a, b]$ for some fixed $m \in (0, 1]$ then the following inequality holds:

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} \int_a^b w(x)dx - \int_a^b f(x)w(x)dx \right| \qquad (2.3) \\ \leq \; & \frac{(b-a)^2}{4} \|w\|_{[0,1],\infty} \min\left\{ |f'(a)| + m \left| f'\left(\frac{b}{m}\right) \right|, m \left| f'(\frac{a}{m}) \right| + |f'(b)| \right\}. \end{aligned}$$

Proof. Let $x \in [a, b]$. Using Lemma 1.6, we have

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} \int_a^b w(x)dx - \int_a^b f(x)w(x)dx \right| \\ \leq \; & \frac{(b-a)^2}{2} \left\{ \int_0^1 \left| \int_0^t w(sa + (1-s)b)ds \right| |f'(ta + (1-t)b)| \, dt \right. \\ & \left. + \int_0^1 \left| \int_t^1 w(sa + (1-s)b)ds \right| |f'(ta + (1-t)b)| \, dt \right\} \\ \leq \; & \frac{(b-a)^2}{2} \|w\|_{[0,1],\infty} \left\{ \int_0^1 t \, |f'(ta + (1-t)b)| \, dt + \int_0^1 |1-t| \, |f'(ta + (1-t)b)| \, dt \right\}. \end{aligned}$$

Since $|f'|$ is $m-$convex on $[a, b]$, we obtain

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} \int_a^b w(x)dx - \int_a^b f(x)w(x)dx \right| \\ \leq \; & \frac{(b-a)^2}{2} \|w\|_{[0,1],\infty} \left\{ \int_0^1 t \left[t \, |f'(a)| + m(1-t) \left| f'\left(\frac{b}{m}\right) \right| \right] dt \right. \\ & \left. + \int_0^1 (1-t) \left[t \, |f'(a)| + m(1-t) \left| f'\left(\frac{b}{m}\right) \right| \right] dt \right\} \\ = \; & \frac{(b-a)^2}{4} \|w\|_{[0,1],\infty} \left\{ |f'(a)| + m \left| f'\left(\frac{b}{m}\right) \right| \right\}. \end{aligned}$$

Analogously we have

$$\left| \frac{f(a) + f(b)}{2} \int_a^b w(x)dx - \int_a^b f(x)w(x)dx \right| \leq \frac{(b-a)^2}{4} \|w\|_{[0,1],\infty} \left\{ m \left| f'\left(\frac{a}{m}\right) \right| + |f'(b)| \right\},$$

which completes the proof. ∎

Theorem 2.5. Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$ and $w : [a, b] \to [0, \infty)$ be a differentiable mapping. If $|f'|^q$ is $m-$convex on $[a, b]$, $q > 1$, for some fixed $m \in (0, 1]$ then the following inequality holds:

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} \int_a^b w(x)dx - \int_a^b f(x)w(x)dx \right| \\ \leq\ & \frac{(b-a)^2}{(p+1)^{\frac{1}{p}}} \|w\|_{[0,1],\infty} \min \left\{ \left[\frac{|f'(a)|^q + m \left|f'\left(\frac{b}{m}\right)\right|^q}{2} \right]^{\frac{1}{q}}, \left[\frac{m\left|f'\left(\frac{a}{m}\right)\right|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}} \right\} \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using Lemma 1.6, Hölder's inequality and the $m-$convexity of $|f'|^q$, for $\frac{1}{p} + \frac{1}{q} = 1$, it follows that

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} \int_a^b w(x)dx - \int_a^b f(x)w(x)dx \right| \\ \leq\ & \frac{(b-a)^2}{2} \left\{ \left(\int_0^1 \left| \int_0^t w(sa+(1-s)b)ds \right|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'(ta+(1-t)b)|^q dt \right)^{\frac{1}{q}} \right. \\ & \left. + \left(\int_0^1 \left| \int_t^1 w(sa+(1-s)b)ds \right|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'(ta+(1-t)b)|^q dt \right)^{\frac{1}{q}} \right\} \\ \leq\ & \frac{(b-a)^2}{2} \|w\|_{[0,1],\infty} \left\{ \left(\int_0^1 t^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \left[t|f'(a)|^q + m(1-t)\left| f'\left(\frac{b}{m}\right)\right|^q \right] dt \right)^{\frac{1}{q}} \right. \\ & \left. + \left(\int_0^1 (1-t)^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \left[t|f'(a)|^q + m(1-t)\left| f'\left(\frac{b}{m}\right)\right|^q \right] dt \right)^{\frac{1}{q}} \right\} \\ =\ & \frac{(b-a)^2}{(p+1)^{\frac{1}{p}}} \|w\|_{[0,1],\infty} \left[\frac{|f'(a)|^q + m\left|f'\left(\frac{b}{m}\right)\right|^q}{2} \right]^{\frac{1}{q}} \end{aligned}$$

and analogously

$$\left| \frac{f(a)+f(b)}{2} \int_a^b w(x)dx - \int_a^b f(x)w(x)dx \right| \leq \frac{(b-a)^2}{(p+1)^{\frac{1}{p}}} \|w\|_{[0,1],\infty} \left[\frac{m\left|f'\left(\frac{a}{m}\right)\right|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}}$$

which completes the proof. ∎

3 Inequalities for quasi-convex functions

Theorem 3.1. Let $f : [0, \infty) \to \mathbb{R}$ be a quasi-convex function, $a, b \in [0, \infty)$ with $a < b$ and $g : [a, b] \to \mathbb{R}$ be nonnegative, integrable and symetric about $\frac{a+b}{2}$. Then

$$\int_a^b f(x)g(x)dx \leq \max\{f(a), f(b)\} \int_a^b g(x)dx.$$

Proof. Since f is quasi-convex and g is nonnegative, integrable and symetric about $\frac{a+b}{2}$, we have

$$\begin{aligned}\int_a^b f(x)g(x)dx &= \frac{1}{2}\left[\int_a^b f(x)g(x)dx + \int_a^b f(a+b-x)g(a+b-x)dx\right] \\ &= \frac{1}{2}\left\{\int_a^b \left[f(x)+f(a+b-x)\right]g(x)dx\right\} \\ &= \frac{1}{2}\left\{\int_a^b \left[f\left(\frac{b-x}{b-a}a+\frac{x-a}{b-a}b\right)+f\left(\frac{x-a}{b-a}a+\frac{b-x}{b-a}b\right)\right]g(x)dx\right\} \\ &\leq \frac{1}{2}\left\{\int_a^b \left[\max\{f(a),f(b)\}+\max\{f(a),f(b)\}\right]g(x)dx\right\} \\ &= \max\{f(a),f(b)\}\int_a^b g(x)dx.\end{aligned}$$

This completes the proof. ∎

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